3459. [2009 : 326, 329] *Proposed by Zafar Ahmed, BARC, Mumbai, India.*

Let a, b, c and p, q, r be positive real numbers. Prove that if $q^2 \le pr$ and $r^2 \le pq$, then

$$\frac{a}{pa+qb+rc} + \frac{b}{pb+qc+ra} + \frac{c}{pc+qa+rb} \le \frac{3}{p+q+r}\,.$$

When does equality hold?

Solution by Arkady Alt, San Jose, CA, USA, expanded by the editor.

Consider the system of linear equations below:

$$\begin{aligned} pa+qb+rc &= x\,, \\ ra+pb+qc &= y\,, \\ qa+rb+pc &= z\,, \end{aligned}$$

The coeffficent matrix has determinant $\Delta = p^3+q^3+r^3-3pqr$. If $\Delta = 0$, then by the AM–GM Inequality $p=q=r$, in which case equality holds in the required inequality.

Otherwise, by Cramer's Rule, we obtain that $a = \frac{1}{\Delta}(x\alpha+y\gamma+z\beta)$, $b = \frac{1}{\Delta}(x\beta+y\alpha+z\gamma)$, and $c = \frac{1}{\Delta}(x\gamma+y\beta+z\alpha)$, where $\alpha = p^2-qr$, $\beta = q^2-rp$, and $\gamma = r^2-pq$.

Thus,

$$\begin{aligned} &\frac{a}{pa+qb+rc} + \frac{b}{pb+qc+ra} + \frac{c}{pc+qa+rb} \\ &= \frac{1}{\Delta}\left(\frac{x\alpha+y\gamma+z\beta}{x} + \frac{x\beta+y\alpha+z\gamma}{y} + \frac{x\gamma+y\beta+z\alpha}{z}\right) \\ &= \frac{3\alpha}{\Delta} + \frac{\beta}{\Delta}\left(\frac{z}{x}+\frac{x}{y}+\frac{y}{z}\right) + \frac{\gamma}{\Delta}\left(\frac{y}{x}+\frac{z}{y}+\frac{x}{z}\right)\,. \qquad (1) \end{aligned}$$

Since x, y, and z are positive, the AM–GM Inequality yields

$$\frac{z}{x}+\frac{x}{y}+\frac{y}{z} \ge 3 \quad \text{and} \quad \frac{y}{x}+\frac{z}{y}+\frac{x}{z} \ge 3\,. \qquad (2)$$

Since $\beta \le 0$ and $\gamma \le 0$ by assumption, we obtain from (1) that

$$\begin{aligned} &\frac{a}{pa+qb+rc} + \frac{b}{pb+qc+ra} + \frac{c}{pc+qa+rb} \\ &\le \frac{3\alpha}{\Delta}+\frac{3\beta}{\Delta}+\frac{3\gamma}{\Delta} = \frac{3(\alpha+\beta+\gamma)}{\Delta} = \frac{3}{p+q+r}\,. \end{aligned}$$

Equality holds if $p=q=r$ or if $\Delta \ne 0$ and $x=y=z$ (necessary for equality to hold in (2)), and in the latter case $a=b=c=\frac{x}{\Delta}(\alpha+\beta+\gamma)$.

Conversely, if $a = b = c$ then equality holds; hence, equality holds if and only if $p = q = r$ or $a = b = c$.

Also solved by MOHAMMED AASSILA, Strasbourg, France; GEORGE APOSTOLOPOULOS, Messolonghi, Greece; OLIVER GEUPEL, Brühl, NRW, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer. There was one incorrect solution submitted.

3460. [2009 : 327, 329] *Proposed by Tran Quang Hung, student, Hanoi National University, Vietnam.*

The triangle ABC has circumcentre O, orthocentre H, and circumradius R. Prove that

$$3R - 2OH \ \leq\ HA + HB + HC \ \leq\ 3R + OH\,.$$

Solution by George Apostolopoulos, Messolonghi, Greece.

We need two known results:

$$\overrightarrow{OH} \ =\ \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}\,, \qquad (1)$$

and, for any three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ in Euclidean space,

$$|\vec{b} + \vec{c}| + |\vec{c} + \vec{a}| + |\vec{a} + \vec{b}| \ \leq\ |\vec{a}| + |\vec{b}| + |\vec{c}| + |\vec{a} + \vec{b} + \vec{c}|\,. \qquad (2)$$

The first is the observation (in vector notation) that the segment joining the orthocentre of a triangle to any vertex (represented, for example, by the vector $\overrightarrow{HA} = \overrightarrow{HO} + \overrightarrow{OA} = \overrightarrow{OA} - \overrightarrow{OH}$) is parallel to and twice as long as the segment joining the midpoint of the opposite side to the circumcentre (namely $-\frac{1}{2}\left(\overrightarrow{OB} + \overrightarrow{OC}\right)$). The second, follows from the triangle inequality applied to Hlawka's identity (see D.S. Mitrinović, *Analytic Inequalities*, Springer, Berlin, 1970, page 171 item 2.25.2, for the proof and for further references).

For the rightmost inequality use (1) together with $HA = |\overrightarrow{OB} + \overrightarrow{OC}|$, $R = |\overrightarrow{OA}|$, and analogous expressions for the other two vertices to rewrite the inequality $HA + HB + HC \leq 3R + OH$ as

$$\begin{aligned} &|\overrightarrow{OB} + \overrightarrow{OC}| + |\overrightarrow{OC} + \overrightarrow{OA}| + |\overrightarrow{OA} + \overrightarrow{OB}| \\ &\qquad \leq\ |\overrightarrow{OA}| + |\overrightarrow{OB}| + |\overrightarrow{OC}| + |\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}|\,, \end{aligned}$$

which holds by (2).

For the inequality on the left, without loss of generality we can assume that $\angle A \leq 60^\circ$. Then $|\overrightarrow{HA}| = 2R\cos A \geq 2R\cos 60^\circ = R$, while

$$|\overrightarrow{HB}| + |\overrightarrow{OH}| \ \geq\ |\overrightarrow{OB}| \ =\ R\,, \quad \text{and} \quad |\overrightarrow{HC}| + |\overrightarrow{OH}| \ \geq\ |\overrightarrow{OC}| \ =\ R\,.$$